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Heat conduction in sliding solids

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Abstract—An examination is made of the conduction between sliding solids with heat energy generated along the region of contact. Based on a Green's function formulation, a Fredholm integral equation of the first kind is derived and an asymptotic solution for the heat flux partition to each solid is obtained for large Peclet numbers. By introducing further asymptotic approximations, closed-form expressions are derived for the temperature fields in the solids. Comparison with a numerical solution indicates that the asymptotic solutions are valid for Peclet numbers greater than ten, which covers most cases of practical interest. In addition, an examination of the present solution reveals the inadequacy of the empirical relations deduced by earlier workers for the estimation of the thermal penetration into the solids. An appropriate parameter for this correlation is suggested.

1. INTRODUCTION

NUMEROUS mechanical processes involve a solid sliding over another with heat energy generated within the contact region. In most cases, the heat transfer around the contact region may be studied by examining the steady-state thermal exchanges between two semi-infinite moving solids with plane boundaries which are perfectly insulated except at the contact region. Two aspects are of special importance in this analysis: partition of the heat energy into each body and the resultant temperature fields in the bodies.

The Peclet number, which depends on the thermal properties and speeds of the solids, and on the contact length, is the major non-dimensional parameter governing the heat transfer mechanism in this system. For very high speeds (hence large Peclet numbers), it has been shown [1] that, in the context of strip rolling, the problem can be further reduced to one of transient one-dimensional heat conduction in stationary bodies, i.e. the thermal diffusion term in the direction of the motion is small compared to the advective component. The solution [1] obtained in this simplified model, however, only provides results within the contact region, and no information can be derived prior to and beyond the contact region.

Most of the other investigations which considered a two-dimensional heat flow in moving bodies involved approximations in predicting the partition of the heat energy to the bodies. Early studies were reported by Blok [2] and Jaeger [3] who expressed the surface temperatures of each solid in terms of the surface heat flux using a Green's function formulation. Instead of determining the heat flux partition by matching the surface temperatures of the solids at all points along the contact region, they estimated an overall heat flux partition by matching either the maximum [2] or average [3] surface temperatures of the solids in addition to assuming a uniform heat flux distribution to each solid. In these studies, the heat source was assumed to be stationary with respect to one of the solids. This same problem was later extended to consider a moving heat source with respect to both solids, with the heat flux partition determined numerically by matching the surface temperatures of the solids using collocation [4, 5].

On the other hand, in the context of thermal damages induced in the grinding process, the temperature field in a moving solid with surface heat flux confined to a finite region has been examined. Nearly all workers have used the Green's function of the temperature field proposed by Jaeger [3] as a starting point. In most cases, numerical integration was performed to determine the entire temperature field with the heat flux distribution assumed to be uniform (e.g. Takazawa [6]) although other distributions have also been examined [7]. Empirical expressions have been deduced from the numerical results [6, 7] as a means of rapid evaluation of the peak temperatures and thermal penetration into the solid. This approach has been widely accepted in evaluating the thermal effects in grinding [8–12].

Barber, in a recent paper [13], discussed the effects of a difference in the bulk temperatures of the solids as well as the subsurface heat generation within the contact region. The analysis was also extended to the case of multiple contacting areas.

More recently, an asymptotic solution was developed in refs. [14, 15], for large Peclet numbers, for a system of sliding bodies in contact as discussed above (with no heat generation along the contact), where the bulk temperatures of the solids are different. The partitioning of the heat flux to each solid was deduced [14] and expressions for the resulting temperature fields of the solids were obtained [15].

In this paper, the thermal exchanges between two

NOMENCLATURE

$f(x)$	$\partial T_1(x,0)/\partial y$, equation (15)
$f_0(x), f_1(x), \ldots$	functions defined in equation (24)
$g_0(x), g_1(x)$	functions defined in equations (17) and (32)
$g_2(u)$	function defined in equation (46)
$I_i[f(x)]$	integral defined in equation (13)
$I_{i1}[f(x)], \ldots, I_{i4}[f(x)]$	integrals defined in equations (19)–(22)
k	thermal conductivity
k_r	ratio of thermal conductivity of body 2 to that of body 1, k_2/k_1, equation (10)
l	contact length
P	Peclet number, $\frac{1}{2}v^0 l/\alpha$, equation (3)
P_r	ratio of Peclet number of body 1 to that of body 2, P_2/P_1, equation (28)
q_0	intensity of the non-dimensionalized uniform heat flux generation along the contact
$q(x)$	non-dimensionalized heat flux generation along the contact, equation (11)
$q^0(x^0)$	heat flux generation along the contact
r	$P(x^2+y^2)^{1/2}$, equation (44)
$T(x,y)$	dimensionless temperature change, $(T^0 - T^0_R)/T^0_R$, equation (1)
$T^0(x^0, y^0)$	temperature
T_{im}	maximum non-dimensionalized temperature reached on the surface of body i, equation (38)
$T_{ip}(y)$	$(T^0_{ip} - T^0_R)/T^0_R$, equation (56)
$T^0_{ip}(y)$	maximum temperature attained at a depth, y, from the surface of body i
T^0_R	bulk temperature
v^0	speed
x, y	Cartesian coordinate pair, nondimensionalized with the contact length, l
x^0, y^0	Cartesian coordinate pair, Fig. 1
x_m	x-coordinate at which the temperature change on the body surface within the contact region is a maximum, equation (39)
$x_p(y)$	x-coordinate at which the temperature change at a depth, y, from the surface is a maximum, equation (54).

Greek symbols

α	thermal diffusivity
β, β_1, β_2	terms defined in equations (27), (34) and (35), respectively
$\varepsilon_1, \varepsilon_2$	small terms defined for integrals of equations (19)–(22)
$\xi(x,y)$	$\frac{1}{2}Py^2/x$, equation (51)
ρ, θ	polar coordinates defined by equations (41) and (42)
$\psi_i(x,y)$	temperature, $T_i(x,y)$, of body i, as a fraction of its maximum temperature, T_{im}, equation (37).

Subscripts (unless defined above)

1	body 1
2	body 2
i	body i.

semi-infinite solids with heat generation at the contact region are examined. A more formal approach in determining the heat flux partition is formulated and the temperature fields in the solids are evaluated. This analysis, although developed in the context of strip rolling, also finds applications in other areas such as thermal considerations in grinding, machining, and highly loaded gear teeth, cams and tappets.

2. PROBLEM FORMULATION

Consider two moving semi-infinite solids with plane boundaries being in perfect contact over a fixed finite region. Let the x^0-axis be aligned with the plane of contact and the solids be moving at uniform speeds, v^0_1 and v^0_2, respectively, in the x^0-direction. The origin of the Cartesian coordinate system (x^0–y^0) is selected at the leading edge of the contact region ($0 < x^0 < l$, where l is the contact length), along which heat energy is generated, as shown in Fig. 1. It is assumed that there is no thermal variation in a direction normal to the x^0–y^0 plane and thus a two-dimensional analysis may be adopted.

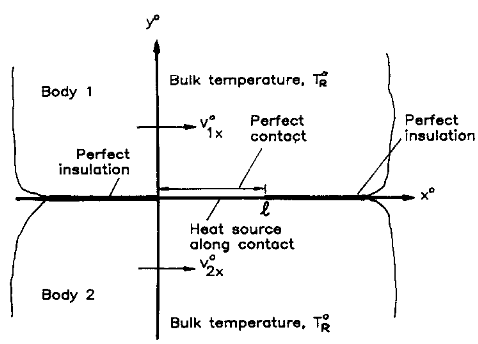

FIG. 1. Thermal system under study.

The temperature fields in the two bodies may be written in terms of their temperature gradients (with respect to y^0) along the contact region based on a Green's function formulation [3, 14], thus

$$T_i(x,y) = \frac{T_i^0(x,y)-T_R^0}{T_R^0} \tag{1}$$

$$= \mp\frac{1}{\pi}\int_0^1 \frac{\partial T_i(x',0)}{\partial y} e^{P_i(x-x')} \times K_0\{P_i[(x-x')^2+y^2]^{1/2}\}\,dx' \tag{2}$$

where

$$P_i = \frac{v_i^0 l}{2\alpha_i} \tag{3}$$

$$x = \frac{x^0}{l} \tag{4}$$

and

$$y = \frac{y^0}{l}. \tag{5}$$

Here subscript $i = 1, 2$, and the upper and lower signs in equation (2), refer to bodies 1 and 2, respectively; $T_i^0(x,y)$ is the temperature of body i; T_R^0 the uniform bulk temperature of the bodies; P_i the Peclet number based on half the contact length; α_i the thermal diffusivity and $K_0()$ is the modified Bessel function of the second kind.

The boundary conditions, in their non-dimensionalized form, are

$$T_1(-\infty,y) = T_2(-\infty,y) = 0 \tag{6}$$

$$\frac{\partial T_1(x,0)}{\partial y} = \frac{\partial T_2(x,0)}{\partial y} = 0 \quad \text{for } x<0,\ x>1 \tag{7}$$

$$T_1(x,0) = T_2(x,0) \quad \text{for } 0<x<1 \tag{8}$$

$$-\frac{\partial T_1(x,0)}{\partial y} + k_r\frac{\partial T_2(x,0)}{\partial y} = q(x) \quad \text{for } 0<x<1 \tag{9}$$

where

$$k_r = \frac{k_2}{k_1} \tag{10}$$

and

$$q(x) = \frac{lq^0(x^0)}{k_1 T_R^0} \tag{11}$$

Here k_i is the thermal conductivity of body i, and $q^0(x^0)$ is the rate of heat generation (per unit area) at the contact region.

The boundary conditions of equations (6) and (7) stipulate that the body temperatures are equal to the bulk temperatures far upstream and that the surface along $y = 0$ is insulated outside the contact region; these are satisfied by equation (2) implicitly as a result of a proper choice of Green's function. Those of equations (8) and (9) specify the conditions of continuity of temperatures along the contact region (perfect contact assumed) and conservation of heat fluxes, respectively.

3. HEAT FLUX ALONG THE CONTACT REGION

On elimination of $\partial T_2(x,0)/\partial y$ from equations (2), (8) and (9), a Fredholm integral equation of the first kind in the unknown function $\partial T_1(x,0)/\partial y$ is obtained

$$I_1[f(x)] + \frac{1}{k_r}I_2[f(x)] = -\frac{1}{k_r}\int_0^1 q(x)\,e^{P_2(x-x')} \times K_0(P_2|x-x'|)\,dx \quad \text{for } 0<x<1 \tag{12}$$

where

$$I_i[f(x)] = \int_0^1 f(x')\,e^{P_i(x-x')}\,K_0(P_i|x-x'|)\,dx' \tag{13}$$

$$= \int_0^x f(x-u)\,e^{P_iu}\,K_0(P_iu)\,du + \int_0^{1-x} f(x+u)\,e^{-P_iu}K_0(P_iu)\,du \tag{14}$$

and

$$f(x) = \frac{\partial T_1(x,0)}{\partial y}. \tag{15}$$

If the heat generation may be assumed uniform, i.e. $q(x) \equiv q_0$, the right-hand side of equation (12) may be integrated readily, giving

$$I_1[f(x)] + \frac{1}{k_r}I_2[f(x)] = g_0(x) \quad \text{for } 0<x<1 \tag{16}$$

where

$$g_0(x) = -\frac{q_0}{k_r}\{x\,e^{P_2x}[K_0(P_2x+K_1(P_2x)] + (1-x)\,e^{-P_2(1-x)}[K_0(P_2\{1-x\}) - K_1(P_2\{1-x\})]\}. \tag{17}$$

An asymptotic solution of $f(x)$ will be derived below for large Peclet numbers, P_1 and P_2 (the Peclet numbers are in the range of 4000 to 60 000 in the case of strip rolling). The kernel of the integral $I_i()$ consists of the modified Bessel function which, although being singular at $x' = x$, decreases rapidly to zero when $x' \neq x$ for large P_i. Thus, a solution which is valid for regions away from the leading and trailing edges of the contact region (i.e. P_ix and $P_i(1-x) \gg 1$) is firstly sought and the correction terms for the small leading and trailing edge regions are then examined.

Consider the region $\varepsilon_1 \ll x \ll (1-\varepsilon_2)$, where $0 < (\varepsilon_1, \varepsilon_2) \ll 1$ but $P_i\varepsilon_1$ and $P_i\varepsilon_2 \gg 1$ ($i = 1, 2$). Following ref. [14], the integrals of equation (14) may be divided up as follows:

$$I_i[f(x)] = I_{i1}[f(x)] + I_{i2}[f(x)] + I_{i3}[f(x)] + I_{i4}[f(x)] \tag{18}$$

where

$$I_{i1}[f(x)] = \int_0^{\varepsilon_1} f(x-u)\,e^{P_i u} K_0(P_i u)\,du \quad (19)$$

$$I_{i2}[f(x)] = \int_{\varepsilon_1}^{x} f(x-u)\,e^{P_i u} K_0(P_i u)\,du \quad (20)$$

$$I_{i3}[f(x)] = \int_0^{\varepsilon_2} f(x+u)\,e^{-P_i u} K_0(P_i u)\,du \quad (21)$$

and

$$I_{i4}[f(x)] = \int_{\varepsilon_2}^{1-x} f(x+u)\,e^{-P_i u} K_0(P_i u)\,du. \quad (22)$$

It is shown in ref. [14] that, for $P_i\varepsilon_1$ and $P_i\varepsilon_2 \gg 1$, the sum of $I_{i1}[f(x)]$ and $I_{i3}[f(x)]$ consists of ε_1- and ε_2-dependent terms only. Thus, equation (16) reduces to

$$I_{12}[f(x)]+I_{14}[f(x)]+\frac{1}{k_r}\{I_{22}[f(x)] + I_{24}[f(x)]\} = g_0(x)+\varepsilon_1,\varepsilon_2\text{-terms}. \quad (23)$$

Now, let

$$f(x) = f_0(x)+f_1(x)+f_2(x)+\cdots \quad (24)$$

where

$$f_i(x) = o[f_{i-1}(x)] \quad \text{for } i = 1,2,3,\ldots \quad (25)$$

By expanding asymptotically the Bessel functions in $I_{i2}[f(x)]$, $I_{i4}[f(x)]$ and $g_0(x)$ for large arguments, and retaining only the leading order terms, equation (23) becomes

$$\int_{\varepsilon_1}^{x} \frac{f_0(x-u)}{u^{1/2}}\,du = -2\beta x^{1/2}+\varepsilon_1\text{-terms} \quad (26)$$

where

$$\beta = \frac{q_0}{1+k_r P_r^{1/2}} \quad (27)$$

and

$$P_r = \frac{P_2}{P_1}. \quad (28)$$

Now, provided that the integral of equation (26) is integrable at the origin, it may be written as the difference between an integral of the integrand from 0 to x and one from 0 to ε_1, the latter obviously consisting only of ε_1-dependent terms which must cancel out with those on the right-hand side of equation (26), thus

$$\int_0^{x} \frac{f_0(x-u)}{u^{1/2}}\,du = -2\beta x^{1/2} \quad (29)$$

which may be solved readily, giving

$$f_0(x) = -\beta. \quad (30)$$

Hence the leading order term of the heat flux to solid 1 along the contact region is a constant, signifying that a uniform heat energy generation would result in a largely uniform heat flux distribution to the two bodies. This result, obtained through a rigorous asymptotic analysis, agrees with that suggested by Blok [2] and Symm [5] who matched only the maximum surface temperatures of the solids.

The integral equation for the higher order of $f(x)$ may be obtained by substituting equation (30) into equation (16) and retaining the leading order terms of the resultant equation, giving

$$I_1[f_1(x)+f_2(x)+\cdots]+\frac{1}{k_r}I_2[f_1(x) + f_2(x)+\cdots] = g_1(x) \quad \text{for } 0<x<1 \quad (31)$$

where

$$\begin{aligned} g_1(x) = \beta\{&x\,e^{P_1 x}[K_0(P_1 x)+K_1(P_1 x)] \\ &+(1-x)\,e^{-P_1(1-x)}[K_0(P_1\{1-x\}) \\ &-K_1(P_1\{1-x\})]\} \\ &-\beta P_r^{1/2}\{x\,e^{P_2 x}[K_0(P_2 x)+K_1(P_2 x)] \\ &+(1-x)\,e^{-P_2(1-x)}[K_0(P_2\{1-x\}) \\ &-K_1(P_2\{1-x\})]\}. \end{aligned} \quad (32)$$

It is obvious from the integral equation (31), which is valid for the entire contact region, and the definition of $g_1(x)$ that $f_n(x) \equiv 0$, for $n = 1,2,3,\ldots$, when $P_r = 1$, i.e. $f_0(x)$ is an exact solution of equation (16) as long as the Peclet numbers of the solids are identical. In addition, it may be shown that $g_1(x)$ is at least $O(P_1^{-1/2})$ of $g_0(x)$, except in regions very close to the entry zone with $P_r \gg 1$. In view of the application intended in this analysis where P_1 and P_2 are large and P_r does not normally vary significantly from unity, the correction term contributed from $g_1(x)$ is not further examined here.

A numerical scheme, which solves an integral equation of the form of equation (12) with a general right-hand side function, has been devised [16] to study heat conduction in sliding bodies with more complicated heat source distributions. Here, the heat flux to solid 1 calculated from equation (30) is compared with the numerical solution in an attempt to examine the regions of validity of the asymptotic solution. It can be seen from the illustration given in Fig. 2 that the two solutions yield excellent agreement when the Peclet numbers of the solids are equal. When $P_1 \neq P_2$, deviations of the asymptotic solution from the numerical solution are observed, especially around the leading and trailing edges of the contact region. These deviations increase as the Peclet number, P_1, of solid 1 becomes small, when the ratio of the Peclet numbers, P_2/P_1, is small or when the ratio of the thermal conductivities, k_r, is large. For the application intended, where P_1 and P_2 are large and k_r and P_r are of order unity, the asymptotic solution of equation (30) is considered satisfactory. In addition, the small

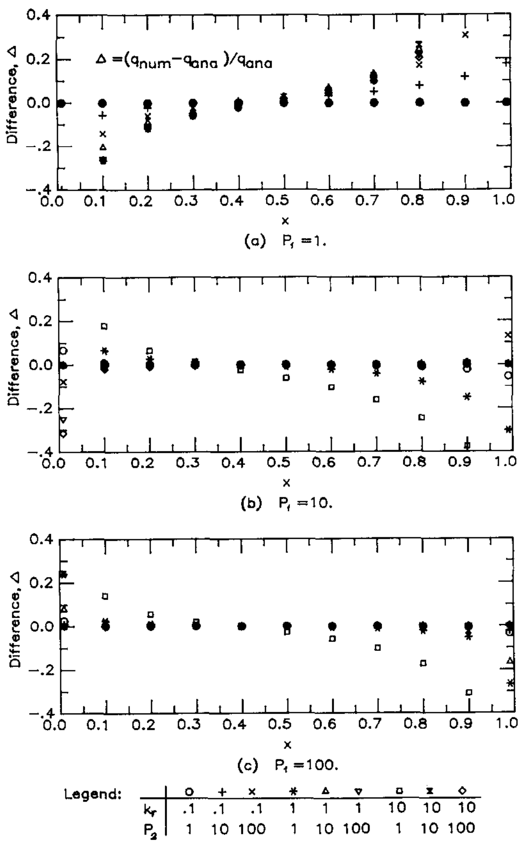

FIG. 2. A comparison of the asymptotic solution with the numerical solution for the heat flux to solid 1.

deviations for the heat flux near the edges of the contact region would not cause significant errors in the subsequent evaluation of the temperatures in the solids, as will be illustrated in the next section.

4. TEMPERATURE DISTRIBUTION

4.1. *Temperatures along the surfaces*

The leading order of the heat flux distribution to body 1, $f_0(x)$, along the contact may be substituted into equation (2) in order to evaluate the temperature fields of the bodies. The integral in equation (2) cannot be evaluated analytically in general, but expressions for the surface temperatures ($y = 0$) may be derived readily, giving

$$T_i(x,0) = \begin{cases} \beta_i\{|x|\,e^{-P_i|x|}[K_1(P_i|x|) - K_0(P_i|x|)] \\ \quad -(1+|x|)\,e^{-P_i(1+|x|)}[K_1(P_i\{1+|x|\}) \\ \quad -K_0(P_1\{1+|x|\})]\} \quad \text{for } x<0 \\ \beta_i\{x\,e^{P_ix}[K_0(P_ix)+K_1(P_ix)] \\ \quad +(1-x)\,e^{-P_i(1-x)}[K_0(P_i\{1-x\}) \\ \quad -K_1(P_i\{1-x\})]\} \quad \text{for } 0<x<1 \\ \beta_i\{x\,e^{P_ix}[K_0(P_ix)+K_1(P_ix)] \\ \quad -(x-1)\,e^{P_i(x-1)}[K_0(P_i\{x-1\}) \\ \quad +K_1(P_i\{x-1\})]\} \quad \text{for } x>1 \end{cases} \tag{33}$$

where

$$\beta_1 = \frac{\beta}{\pi} = \frac{q_0}{\pi(1+k_rP_r^{1/2})} \tag{34}$$

and

$$\beta_2 = \frac{\beta P_r^{1/2}}{\pi} = \frac{q_0P_r^{1/2}}{\pi(1+k_rP_r^{1/2})}. \tag{35}$$

Here $K_1()$ is the modified Bessel function of the second kind of first order.

For regions remote from the leading and trailing edges of the contact (more precisely, when $P_i|x|$ and $P_i|1-x| \gg 1$), equations (33) may be further simplified, giving

$$T_i(x,0) \approx \begin{cases} \dfrac{\beta_i}{2P_i}\left(\dfrac{\pi}{2P_i|x|}\right)^{1/2} e^{-2P_i|x|} \\ \quad \text{for } P_ix \ll -1 \\ \beta_i\left(\dfrac{2\pi x}{P_i}\right)^{1/2} \quad \text{for } P_ix \gg 1;\ P_i(1-x) \gg 1 \\ \beta_i\left(\dfrac{2\pi}{P_i}\right)^{1/2}[x^{1/2}-(x-1)^{1/2}] \\ \quad \text{for } P_i(x-1) \gg 1. \end{cases} \tag{36}$$

A comparison of the analytical solution of equations (33) with the numerical solution of ref. [16], as shown in Fig. 3, produces excellent agreement for a wide range of k_r and P_r values as long as the Peclet numbers are reasonably large. In this illustration, the non-dimensionalized surface temperature has been normalized with respect to the asymptotic peak temperature (also known as the flash temperature [17] in the context of scuffing and wear), T_{im}, thus

$$\psi_i(x,y) = \frac{T_i(x,y)}{T_{im}} \tag{37}$$

where

$$T_{im} = \beta_i\left(\frac{2\pi}{P_i}\right)^{1/2} \tag{38}$$

such that $\psi_i \leqslant 1$. It can be seen that the peak temperature on the surface is located near the trailing edge of the contact region when the Peclet number is large, and that it shifts back towards the centre of the contact region as P_1 is reduced to unity, with its value being smaller than the asymptotic prediction. Hence equations (33) (and equations (36) when applicable), which provide closed-form expressions for the surface temperatures of the solids, are valid for most applications. The peak temperature in the solid, T_{im}, can be shown to occur on the body surface within the contact region, thus it can be readily evaluated from equations (33) after its location, x_m, is determined. It is found, by setting $\partial T_i(x,0)/\partial x = 0$ in the second of equations (33), that x_m is the solution of the equation:

$$e^{P_i}K_0(P_ix_m) = K_0[P_i(1-x_m)]. \tag{39}$$

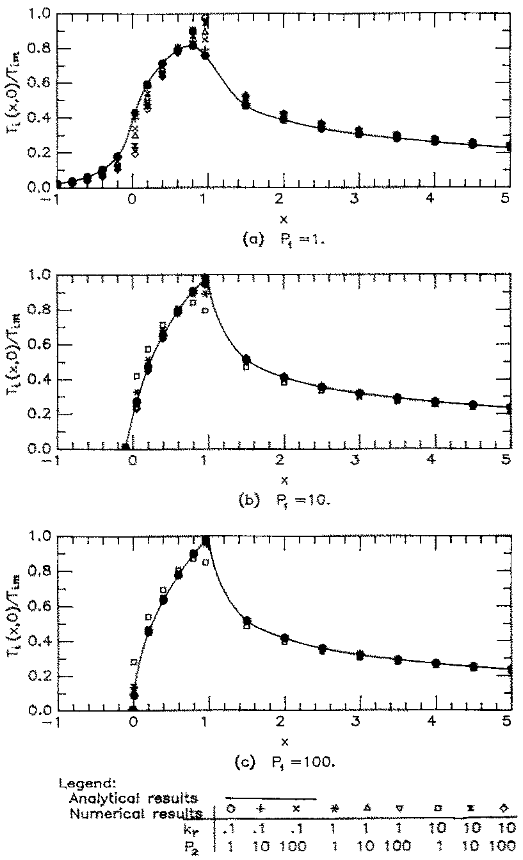

FIG. 3. A comparison of the asymptotic solution with the numerical solution for the body surface temperature.

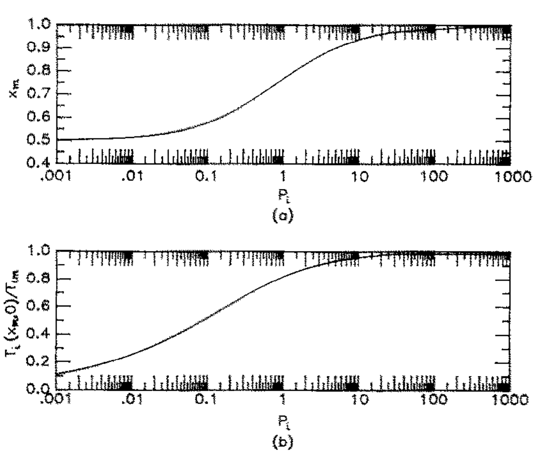

FIG. 4. Variation of (a) the location, x_m, of the peak temperature on the surface, and (b) the associated value, $T_i(x_m, 0)$, with the Peclet number, P_i, of the solid.

A plot of the variation of x_m with P_i and the associated peak temperature is shown in Fig. 4. When the Peclet number is extremely small (note that in this case P_r should be very close to unity for the solution to be valid), the maximum temperature occurs at the mid-point of the contact with a value well below the asymptotic prediction. As P_i increases, x_m moves towards the trailing edge and the asymptotic value is approached. This observation has also been discussed by previous workers (e.g. Jaeger [3]).

4.2. *Temperature field of the entire body*

The temperature field of body i, based on the leading order of the heat flux distribution, $f(x)$, is, from equation (2)

$$T_i(x, y) = \beta_i \int_0^1 e^{P_i(x-x')} \times K_0\{P_i[(x-x')^2+y^2]^{1/2}\}\, dx'. \quad (40)$$

Approximations to the above integral are now introduced in the various regions in order to obtain closed-form expressions. In view of the insignificant temperature change prior to the contact region ($x < 0$) for large Peclet numbers (Fig. 3), details of the temperature in this region will not be sought here.

(a) Within the contact region ($0 < x < 1$). Let

$$x = \rho \cos\theta \quad (41)$$

and

$$y = \rho \sin\theta \quad (42)$$

such that $0 \leqslant |\theta| \leqslant \frac{1}{2}\pi$ and the temperature field is then given by

$$T_i(x, y) = \beta_i \rho \int_0^{1/\rho} e^{-r_i(u-\cos\theta)} \times K_0[r_i(u^2-2u\cos\theta+1)^{1/2}]\, du \quad (43)$$

where

$$r_i = P_i \rho \quad (44)$$

and

$$\rho = (x^2+y^2)^{1/2}. \quad (45)$$

Reference [15] has shown that since r_i is generally large and $|\theta| \ll 1$ in the region of interest (the thermal penetration in the y-direction is of the order of $P_i^{-1/2}$ [1]), the integrand in equation (43) may be approximated by

$$g_2(u) = \begin{cases} \left(\dfrac{\pi}{2r_i}\right)^{1/2} \dfrac{\exp[-2r_i \sin^2(\frac{1}{2}\theta)/(1-u)]}{(1-u)^{1/2}} & \text{for } 0 < u < 1 \\ 0 & \text{for } u > 1. \end{cases} \quad (46)$$

Thus

$$T_i(x, y) \approx \beta_i \left(\frac{\pi\rho}{2P_i}\right)^{1/2} \times \int_0^1 \frac{\exp[-2r_i \sin^2(\frac{1}{2}\theta)/(1-u)]}{(1-u)^{1/2}}\, du \quad (47)$$

which can be integrated after a substitution $w = (1-u)^{-1}$ is made, giving

$$T_i(x,y) = \pi\beta_i\left(\frac{2\rho}{P_i}\right)^{1/2} \operatorname{ierfc}\,[2P_i\rho\sin^2(\tfrac{1}{2}\theta)]^{1/2} \quad (48)$$

where ierfc( ) is the repeated integral of the error function.

In the region of interest, $|y| \ll x$ and equation (48) can be further simplified, giving

$$T_i(x,y) \approx \pi\beta_i\left(\frac{2x}{P_i}\right)^{1/2} \operatorname{ierfc}\left(\frac{P_i y^2}{2x}\right)^{1/2}. \quad (49)$$

The above solution agrees with that of ref. [1] in which the thermal diffusion along the direction of body motion is ignored. It is also noted that equation (49) degenerates to equation $(36)_2$ when $y = 0$.

(b) Beyond the contact region ($x > 1$). For regions away from the trailing edge of the contact ($P_i(x-1) \gg 1$), the modified Bessel function in the integral of equation (40) may be expanded for large argument, and in the region of interest ($|y| \ll x$) with $P_i \gg 1$, the integrand may be approximated and equation (40) is reduced to

$$T_i(x,y) \approx \beta_i\left(\frac{\pi x}{2P_i}\right)^{1/2} e^{-\xi_i}\int_0^{1/(x-1)} \frac{e^{-\xi_i u}}{(1+u)^{3/2}}\,du \quad (50)$$

where

$$\xi_i \equiv \xi_i(x,y) = \frac{P_i y^2}{2x}. \quad (51)$$

The integral in equation (50) may be readily evaluated, giving

$$T_i(x,y) = \pi\beta_i\left(\frac{2x}{P_i}\right)^{1/2}\left[\operatorname{ierfc}(\xi_i^{1/2}) - \left(\frac{x-1}{x}\right)^{1/2}\operatorname{ierfc}\left(\xi_i\frac{x}{x-1}\right)^{1/2}\right]. \quad (52)$$

It is obvious that equation (52) degenerates to equation $(36)_3$ when $y = 0$ (thus $\xi_i = 0$). In addition, it agrees with equation (49) at the trailing edge of the contact region ($x = 1$).

The entire temperature field as calculated from the asymptotic solutions (equations (48) and (52)) has been compared and found to agree well with the numerical solution of ref. [16] for a wide range of k_r and P_r values and for Peclet numbers reduced to as low as 10. An illustration of the comparisons is given in Fig. 5. For low Peclet numbers, slight deviations are observed at the leading and trailing edges of the contact region. These arise since the thermal diffusion in the x-direction becomes important when the Peclet numbers are small (the diffusion term becomes significant compared to the advective term in the regions $0 \leqslant P|x| \leqslant O(1)$ and $0 \leqslant P|1-x| \leqslant O(1)$, respectively).

The comparison shown in Fig. 5 also reinforces the suggestion made earlier that, for high Peclet numbers, retention of only the leading order term of the heat

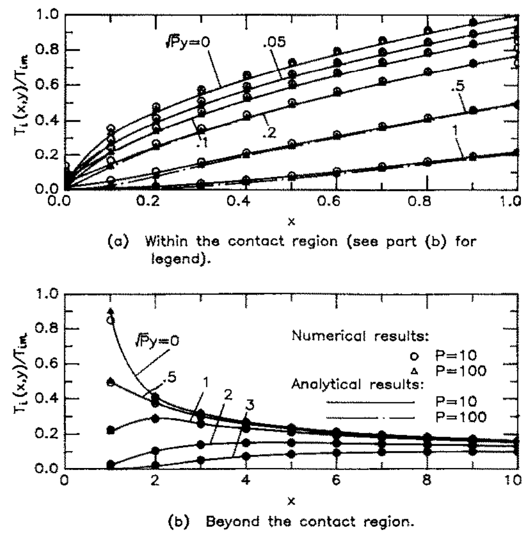

FIG. 5. A comparison of the analytical solution with the numerical solution for the temperatures in the body ($P_1 = P_2 = P$).

flux distribution to the bodies is sufficient in calculating the temperature field.

Of interest is the thermal penetration below the surface of the solids. It has been demonstrated earlier that the surface temperature reaches a maximum within the contact zone and its location approaches the trailing edge ($x = 1$) when the Peclet numbers are large. From Fig. 5, it is obvious that the maximum temperature that a point at a specified depth below the surface would reach will be located beyond the contact region when the Peclet number is large. This location, x_p, may be determined by finding, for a given y, a local maximum of the temperature with respect to x. Thus, from equation (52)

$$\frac{\partial T_i}{\partial x} = \beta_i\left(\frac{2\pi}{P_i}\right)^{1/2}\left\{\frac{\exp(-\xi_i)}{2x^{1/2}} - \frac{\exp\left(-\xi_i\frac{x}{x-1}\right)}{2(x-1)^{1/2}}\right\} \quad (53)$$

which, when set to zero, yields an equation for the solution of x_p

$$x_p(x_p-1)\ln\left(\frac{x_p}{x_p-1}\right) = Py^2. \quad (54)$$

The variations of the maximum temperature and its corresponding location with the depth below the surface are given in Fig. 6. It can be seen that the thermal penetration is inversely proportional to $P_i^{1/2}$, thus the higher the speed of the solid, the smaller are the thermal effects (relative to the surface temperature) below the surface.

Similar studies have been carried out in relation to the grinding process by other workers. With numerical integration of equation (40) assuming a uniform heat flux distribution to a solid, empirical expressions

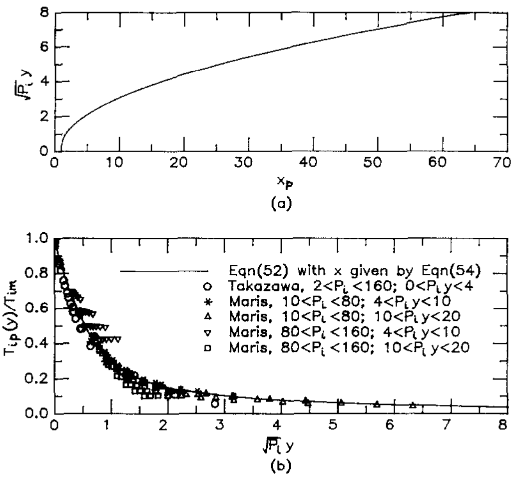

FIG. 6. Variation of (a) the location, x_p, of the maximum temperature, and (b) the associated value, $T_{ip}(y)$, with the depth below the surface, y.

for the maximum temperatures, T_{ip}, were fitted in terms of the Peclet number, P_i, for a specified depth, y, below the surface. On writing those results in the present notation, we obtain, from Takazawa [6]

$$\frac{\pi k_i v_i}{2q_i^0 \alpha_i}(T_{ip}^0 - T_R^0) \equiv \frac{P_i}{\beta_i} T_{ip}$$

$$= 3.1(\tfrac{1}{2}P_i)^{0.53} \exp[-0.69(\tfrac{1}{2}P_i)^{-0.37}(P_i y)]$$

$$\text{for } 2 < P_i < 160 \text{ and } 0 < P_i y < 4 \quad (55)$$

where q_i^0 is the rate of heat transfer to body i along the contact region, T_{ip}^0 the maximum temperature reached, and T_{ip} the non-dimensionalized temperature given by

$$T_{ip} = \frac{T_{ip}^0 - T_R^0}{T_R^0} \quad (56)$$

and, from Maris [7], as quoted by Snoeys *et al.* [10]

$$\frac{P_i}{\beta_i} T_{ip} = \begin{cases} 0.66(\tfrac{1}{2}P_i)^{0.93} \exp[-0.15(\tfrac{1}{2}P_i)^{-0.06}(P_i y)] \\ \quad \text{for } 10 < P_i < 80 \text{ and } 4 < P_i y < 10 \\ 0.355(\tfrac{1}{2}P_i)^{1.01} \exp[-0.08(\tfrac{1}{2}P_i)^{0.044}(P_i y)] \\ \quad \text{for } 10 < P_i < 80 \text{ and } 10 < P_i y < 20 \\ 1.64(\tfrac{1}{2}P_i)^{0.67} \exp[-0.0288(\tfrac{1}{2}P_i)^{0.244}(P_i y)] \\ \quad \text{for } 80 < P_i < 160 \text{ and } 4 < P_i y < 10 \\ 0.079(\tfrac{1}{2}P_i)^{1.43} \exp[-0.0187(\tfrac{1}{2}P_i)^{0.44}(P_i y)] \\ \quad \text{for } 80 < P_i < 160 \text{ and } 10 < P_i y < 20. \end{cases} \quad (57)$$

Here T_{ip} is the maximum temperature reached for a point at a distance y below the surface. In particular, the third coefficient in equation $(57)_3$ has been changed to 0.0288, instead of the value of 0.288 as quoted by Snoeys *et al.* [10], in order for a reasonable agreement with the present solution to be obtained.

The above expressions are also plotted in Fig. 6 within their regions of validity for comparison, and

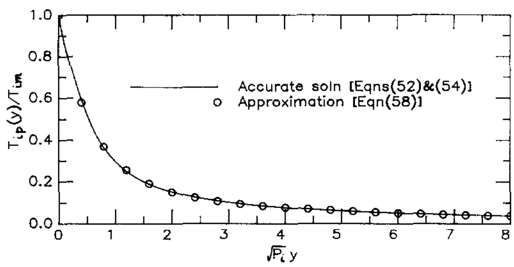

FIG. 7. A comparison of the empirically derived maximum temperatures with those from the asymptotic solution.

generally good agreement is obtained. The slight discrepancies observed in these comparisons could be largely due to inaccuracies involved in the numerical integration and curve fitting by a limited number of expressions in the previous works [6, 7]. Thus, the solution presented here provides a more logical approach in estimating the maximum temperature in the solid: the location, x_p, of the maximum temperature is determined from equation (54), followed by calculating its value from equation (52). Moreover, if it were desirable to develop an empirical expression for rapid determination of these peak temperatures, it is obvious that the peak temperature should be related to the parameter $P_i^{1/2}y$. Only a single expression is then sufficient to cover a wide range of P_i and y (cf. relations each deduced for a narrow region of validity previously [6, 7]). One such expression is

$$\psi_{ip} \equiv \frac{T_{ip}}{\beta_i\left(\dfrac{2\pi}{P_i}\right)^{1/2}} \equiv \left(\frac{k_i}{2q_i^0}\right)\left(\frac{\pi v_i}{\alpha_i l}\right)^{1/2}(T_{ip}^0 - T_R^0)$$

$$= \exp[-1.507P_i^{1/2}y + 0.3610(P_i^{1/2}y)^2 - 0.0446(P_i^{1/2}y)^3 + 0.00208(P_i^{1/2}y)^4] \quad (58)$$

for $0 \leqslant (P_i^{1/2}y) \leqslant 8$. A comparison of equation (58) with the asymptotic solution is shown in Fig. 7. Indeed, excellent agreement is obtained.

5. CONCLUSION

The conduction of heat in sliding solids with heat energy generation along the contact region is examined in this paper. An asymptotic solution for large Peclet numbers is derived for the heat flux partition to the solids. It is demonstrated that, with a uniform heat source, the heat flux partition to the solids is essentially uniform and is governed by the parameter $kP^{1/2}$ of each solid (where k is the thermal conductivity and P the Peclet number).

On introducing further approximations for large Peclet numbers, closed-form expressions for the temperature fields in the solids are deduced. Peak temperatures at specified positions below the solids are also predicted and an appropriate parameter, which offers much improvement over those suggested by previous workers, for the correlation of the peak temperatures is suggested.

The asymptotic solutions obtained in this paper have been checked with a numerical solution, and found to be valid for Peclet numbers greater than 10. While this study has been carried out in the context of strip rolling in which the Peclet numbers are extremely high (in the range 4000 to 60 000), the solutions can also be applied to the examination of thermal effects in other relevant processes such as grinding, machining, rubbing in gear teeth and cams, in which cases the Peclet numbers are usually higher than 10.

Similar thermal problems arising from certain mechanical processes can be formulated and an asymptotic solution deduced with the technique discussed in this paper. The more notable ones are the heat transfer in sliding bodies where the two bodies have different bulk temperatures (this problem has been discussed in ref. [14]) and/or where heat energy is generated in one or both bodies due to deformation. An associated problem, where the bodies move in opposite directions relative to the contact area, for which the grinding process is a classical example, can be formulated in a similar manner. However, the asymptotic analysis following the same procedure discussed in this paper leads to an integral equation having a form which would need to be solved numerically or by a technique involving complex analysis.

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CONDUCTION THERMIQUE DANS DES SOLIDES GLISSANTS

Résumé—On considère la conduction entre des solides glissants avec la formulation de chaleur le long de la surface de contact. A partir de la formulation de la fonction de Green, on obtient une équation intégrale de Fredholm de première espèce ainsi qu'une solution asymptotique pour la partition entre chaque solide est obtenue pour de grands nombres de Péclet. En introduisant des approximations asymptotiques, des expressions exactes sont dérivées pour le champ de température dans les solides. Une comparaison avec une solution numérique indique que les solutions asymptotiques sont valables pour des nombres de Péclet supérieur à 10, ce qui couvre la plupart des cas d'intérêt pratique. De plus, un examen de la présente solution révèle l'inadéquation des relations empiriques déduites par les premiers chercheurs pour l'estimation de la pénétration thermique dans les solides. On suggère un paramètre approprié pour cette corrélation.

WÄRMELEITUNG IN ÜBEREINANDER GLEITENDEN FESTKÖRPERN

Zusammenfassung—Es wurde eine Untersuchung der Wärmeleitung in übereinander gleitenden Festkörpern durchgeführt, in deren Kontaktbereich Wärme freigesetzt wird. Basierend auf den Green'schen Funktionen wurde eine Fredholm-Integralgleichung erster Ordnung hergeleitet und eine asymptotische Lösung für den jeweiligen Anteil des Wärmestroms in die beiden Körper für große Peclet-Zahlen ermittelt. Durch Einführen zusätzlicher asymptotischer Näherungen wurden Ausdrücke in geschlossener Form für das Temperaturfeld in den Festkörpern abgeleitet. Der Vergleich mit einer numerischen Lösung deutet darauf hin, daß die asymptotischen Lösungen für Peclet-Zahlen größer als 10 gültig sind, was die meisten Fälle des praktischen Interesses abdeckt. Zusätzlich offenbart eine Überprüfung mit der vorliegenden Lösung die Unzulänglichkeit der empirischen Beziehungen, die in vorangegangenen Arbeiten für das Eindringen von Wärme in die Körper hergeleitet wurden. Für die Korrelation wird ein geeigneter Parameter vorgeschlagen.

ТЕПЛОПРОВОДНОСТЬ ТВЕРДЫХ ТЕЛ ПРИ СКОЛЬЖЕНИИ

Аннотация—Проведено исследование передачи тепла теплопроводностью между телами при скольжении, когда в области их контакта выделяется тепловая энергия. С помощью метода функций Грина выведено интегральное уравнение Фредгольма первого рода, и для больших чисел Пекле получено асимптотическое решение, определяющее распределение потока тепла в каждом из тел. С использованием асимптотических приближений получены в замкнутом виде вырахения для распределений температур в твердых телах. Сравнение с численным решением показывает, что асимптотическое описание справедливо для значений числа Пекле, превышающих 10, т.е. тех значений, которые чаще всего встречаются на практике. Кроме того, анализ полученного решения позволил выявить неадекватность ранее предложенных эмпирических соотношений, использовавшихся для оценки распределения тепла в контактирующих телах. Для корреляции предложен соответствующий параметр.